

The "Dilemma" of light springs

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Abstract: Starting from the mechanical and energy characteristics of the light spring physical model, this paper highlights the potential dilemmas that may arise when applying the light spring model in problem-solving. It then separately demonstrates the causes of these dilemmas and the corresponding methods to resolve them from the perspectives of both classical mechanics and special relativity.

Keywords: Light spring, Energy dilemma, Classical mechanics, Special relativity

1. Introduction

A light spring (hereafter referred to as a "light spring")—that is, a spring with zero mass and whose elastic force obeys Hooke's law in relation to deformation—is a common idealized model in middle school physics problem-solving. Simplifying an ordinary spring to a light spring can greatly facilitate problem-solving. However, this seemingly simple model itself contains certain "dilemmas," which this paper attempts to briefly analyze.

2. Mechanical characteristics of the light spring

Due to its zero mass, according to Newton's second law, the forces at both ends of a light spring must be equal in magnitude and opposite in direction.

[Example 1] As shown in Figure 1, a heavy object m is suspended from a light spring, which is in turn fixed to the ceiling with a thin rope. After the entire apparatus is balanced and stationary, the rope is burned through with a flame. At the instant the rope breaks, the acceleration of m is a (ignoring air resistance). Then:

- A. $a < g$, direction downward
- B. $a = g$, direction downward
- C. $a > g$, direction downward
- D. $a < g$, direction upward

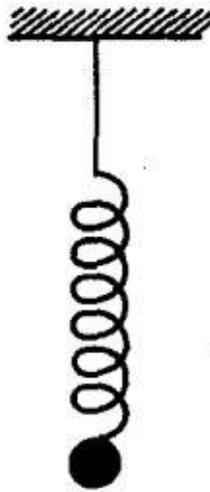


Figure 1

At the moment the thin rope is cut, the force acting on the upper end of the spring instantly becomes zero. Since the net force on a light spring must always be zero, the force at its lower end also becomes zero instantaneously. Therefore, the spring's tension on the object also instantly drops to zero, leaving the object subject only to gravity. As a result, its acceleration equals g .

3. Energy characteristics of the light spring

When the ends of a light spring are connected to objects with mass, the system can store elastic potential energy. Taking the elastic potential energy of the spring at its natural length as zero, within the elastic limit, the relationship between elastic potential energy and the deformation of the spring satisfies:

$$E_p = \frac{1}{2}kx^2$$

where E_p is the elastic potential energy of the system, k is the spring constant, and x is the deformation of the spring.

[Example 2] As shown in Figure 2, a small ball falls freely from rest at point a , makes contact with the spring at point b , and compresses the spring to its shortest length at point c . If the mass of the spring and air resistance are neglected, during the motion of the ball from $a \rightarrow b \rightarrow c$:

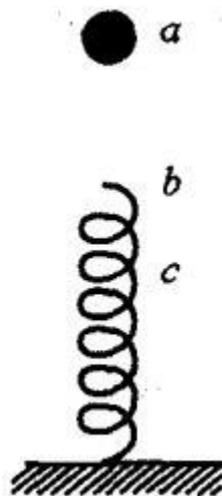


Figure 2

- A. The total mechanical energy of the ball and the spring is conserved.
- B. The gravitational potential energy of the ball decreases uniformly over time.
- C. The kinetic energy of the ball is maximum at point b.
- D. The decrease in the gravitational potential energy of the ball when it reaches point c equals the increase in the elastic potential energy of the spring.

The correct answers are A and D.

Detailed explanation: Taking the system composed of the ball and the spring as the subject of study, during the motion of the ball, only gravity and the spring's elastic force do work, satisfying the condition for conservation of mechanical energy. Therefore, the mechanical energy of the system is conserved, making option A correct. During the ball's descent, only gravity and the spring's elastic force do work. When the ball reaches point c, the decrease in its gravitational potential energy equals the increase in the spring's elastic potential energy, making option D correct.

The explanations similar to the reference answers above are very common in many reference books. However, phrases such as "the gravitational potential energy of the ball" and "the elastic potential energy of the spring" are inherently problematic. The "gravitational potential energy of the ball" actually refers to the gravitational potential energy of the system composed of the ball and the Earth. Without the Earth, gravitational potential energy would be meaningless. Similarly, the "elastic potential energy of the spring" actually refers to the elastic potential energy of the system composed of the ball, the spring, and the Earth. Without the ball or the Earth, the elastic potential energy of the spring would also be meaningless. In most cases, when referring to the gravitational potential energy of an object as described in a problem, we interpret it as the gravitational potential energy of the system formed by the object and the Earth. However, if the elastic potential energy of the spring is understood as the energy stored in the spring itself or as the energy shared by the spring and one of the objects connected to it, such interpretations are incorrect and can lead to "dilemmas."

4. The "Dilemma" of the light spring

Dilemma 1: The Issue of Sudden Changes in Mechanical Energy

[Example 3] The scenario is the same as in Example 1. Discuss the changes in the mechanical energy of the system composed of the ball and the spring before and after the rope is cut.

Let us first analyze this from the perspective of the elastic potential energy of the spring or the elastic potential energy of the system composed of the spring and the ball. This is, of course, a common way of understanding it:

It is easy to see that at the moment the rope breaks, both the position and velocity (which is zero) of the ball remain unchanged, so the mechanical energy of the system (composed of the ball and the Earth) remains constant. However, before the rope breaks, the spring has tensile deformation. Let its spring constant be k , then the elastic potential energy of the system (composed of the spring and the ball) is $\frac{m^2 g^2}{2k}$. After the rope breaks, since the forces at both ends of the light spring instantly reduce to zero, meaning the deformation of the spring becomes zero at that moment, the elastic potential energy of the system (composed of the spring and the ball) also instantly reduces to zero.

The dilemma then arises: where has the vanished elastic potential energy gone?

Dilemma 2: The Failure of the Work-Energy Principle

[Example 4] As shown in Figure 3, an object with mass $M=2\text{kg}$ is acted upon by a horizontal constant force $F = 30\text{N}$ applied through a light spring on its left end. If the spring constant is $k=100\text{N/m}$ and the coefficient of kinetic friction between the object and the horizontal surface is $\mu=0.3$, find the velocity of the object after it starts moving from rest over a distance of 2m . (Take $g = 10\text{m/s}^2$)



Figure3

Solution 1: Taking the spring and the object as a combined system, according to the work-energy principle (the total work done by external forces and non-conservative internal forces on the system equals the change in the system's mechanical energy), the total work done by the constant force F and the kinetic friction equals the change in the system's mechanical energy. That is:

$$F(s + \Delta x) - \mu Mgs = \frac{1}{2}Mv^2 + \frac{1}{2}k\Delta x^2$$

where S is the displacement of the object, and Δx is the deformation of the spring. Substituting the numerical values yields:

$$v = 7.25\text{m/s}$$

Solution 2: Since the forces at both ends of a light spring must be equal in magnitude, the work-energy theorem can be applied to the object:

$$Fs - \mu Mgs = \frac{1}{2}Mv^2$$

After substituting the numerical values, we obtain:

$$v = 6.93\text{m/s}$$

Thus, a dilemma arises: neither solution is incorrect, so what causes the discrepancy in the results?

Dilemma 3: The Dilemma of Unreasonable Selection of the Study Object

[Example 5] A small ball of mass M is connected to a spring with a spring constant K to form a system, placed on a smooth horizontal table. The other end of the spring is fixed to a wall, and the ball undergoes one-dimensional free vibrations. In a reference frame moving uniformly along the length of the spring with velocity U , is the mechanical energy of this system conserved? Provide reasoning.

The reference answer to this question has sparked considerable controversy among educators. Journals such as *Physics Teacher* and *Physics Bulletin* have published discussions by teachers specifically addressing this question [1-4]. Teachers arguing for conservation and those arguing against it each have their own justifications, and no consensus has been reached.

The dilemma arises: From the perspective of mathematical derivation, the reasoning of several teachers is correct. So, is the mechanical energy of this system conserved or not?

5. The seemingly valid yet problematic "Dilemma"

5.1. How to understand a light spring

From a mechanical perspective, a light spring can provide a linear restoring force to the objects connected to it. For this reason, the light spring can be imagined as a kind of "field," similar to a gravitational or electric field. The difference lies in that the latter two provide a restoring force that follows an inverse-square law with distance. Another important distinction is that a light spring is massless, whereas both gravitational and electric fields possess mass. This raises the question: Can a massless field exist? Or, can a massless object (or substance) exist? If an object (or substance) has no mass, can it still be called an object (or substance)? How can we perceive the existence of a massless object?

In the case of the light spring, perhaps the last question is easy to answer: We can perceive its existence through the linear restoring force it provides. Thus, as seen in Example 1, the ability of a light spring to

provide a linear restoring force is conditional. It can only do so when both ends are connected to objects with mass (before the string is burned). If either end is disconnected (after the string is burned), the force on the light spring immediately vanishes (drops to zero).

The accumulation of force over space is work, and work measures the transformation or transfer of energy. Therefore, once the force on the light spring disappears, its ability to do work is immediately lost, and energy becomes a moot point. In other words, discussing the elastic potential energy of a spring is also conditional, and the condition is the same as that for a light spring to provide a linear restoring force—both ends of the light spring must be connected to objects with mass. Given this, as discussed in Example 2, what we commonly refer to as the "elastic potential energy of the spring" is neither the "elastic potential energy of the light spring itself" nor the "elastic potential energy of the spring and the ball (one of the connected objects)." Instead, it is the "elastic potential energy of the system composed of the spring and the objects connected to both ends."

5.2. Resolving the "Dilemma"

With the above understanding, the "dilemmas" presented in Examples 3 to 5 can be easily resolved.

In Example 3, the system possessed elastic potential energy before the string was cut. After the string was cut, the prerequisite for discussing elastic potential energy no longer existed.

In Example 4, Solution 1 discussed the elastic potential energy of the system. However, since the light spring was connected to an object at only one end, the prerequisite for discussing elastic potential energy was not satisfied. Therefore, this approach was incorrect. In fact, for an abstract force to act on a spring, it must be applied by an object with mass (gravitational and electromagnetic fields, for instance, also possess mass). In other words, to realize the scenario in Example 4, the left end of the spring must also be connected to an object with non-zero mass. For example, if the left end is connected to a smooth (or non-smooth) small object with a mass of 0.01 kg (or any non-zero mass), then whether using Solution 1 or Solution 2, the velocity obtained by object M would be 6.93 m/s.

Similarly, in Example 5, discussing the mechanical energy of the system consisting of the light spring and the ball while disregarding the fixed wall is equivalent to discussing only the kinetic energy of the ball. Of course, it is not conserved, and there is little meaningful discussion to be had. If the wall is also included in the system consisting of the light spring and the ball, the mechanical energy is undoubtedly conserved—though this hardly warrants further discussion.

5.3. The perspective of relativity

The principle of relativity in special relativity tells us that the laws of physics have the same mathematical form in all inertial frames of reference. In other words, all inertial frames are equivalent for describing physical phenomena. Therefore, in Example 5, if the mechanical energy of the system is conserved in the ground frame (an inertial frame), there is no reason for it not to be conserved in a frame moving uniformly along the length of the spring with velocity u (also an inertial frame). From this perspective, the reference answer to this question is indeed incorrect.

According to the mass-energy equivalence equation of special relativity, $E = mc^2$, if $m = 0$, then E must also be zero. Since a light spring has no mass, it inherently cannot store energy. Only when connected to objects with mass does the possibility of storing energy arise. Therefore, "the elastic potential energy of the spring" must refer to "the elastic potential energy of the system consisting of the spring and the objects it is connected to." Without the connected objects, elastic potential energy becomes meaningless—a conclusion consistent with our earlier analysis from the perspectives of force, work, and energy in "How to Understand a Light Spring."

The mass-energy equation also reveals that the idealized model of a "light spring" is inherently flawed. An object with zero rest mass must travel at the speed of light (like photons). An object with zero rest mass but a finite, non-relativistic speed cannot exist in physics. In other words, a light spring is not a physical object in the realm of physics.

In fact, the light spring model inherently contains a self-contradiction: it has zero mass yet can interact with objects. This contradiction is particularly evident in Example 4. The reason for the failure of the work-energy principle in this case lies in the process of the spring's deformation changing from zero to Δx .

During this process, if a constant force F is applied to the left end of the spring, the spring would violate Hooke's Law. Therefore, the assumption in the problem statement that a constant force F can be applied is inherently self-contradictory. Consequently, any conclusion derived from this assumption based on the work-energy principle is naturally flawed. However, once the spring has non-zero mass, applying a constant force F becomes feasible, and the work-energy principle no longer fails.

Nevertheless, as an idealized model, the light spring greatly simplifies the treatment of many problems, especially in mechanics, and serves as a very good approximation for practical scenarios. Thus, the existence of the light spring is not only necessary but also meaningful.

6. Conclusion

When discussing the energy characteristics of a light spring, it is imperative not to disregard the objects connected to its two ends. This is a fundamental point that educators who design exam questions must first clarify—otherwise, dilemmas may arise. These could manifest as the sudden change in energy seen in Example 3, the failure of the work-energy principle as observed in Example 4, or the dilemma stemming from an unreasonable choice of the study object, as illustrated in Example 5.

7. References

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