Teaching mathematical concepts based on the principle of "Consistency in Explanation"

Zhang Guochao[1, a, *], **Zhan Xu**[1, b],

[1]Jiangxi Normal University, No. 99 Ziyang Avenue, Nanchang City, Jiangxi Province, 330022, China
a.zguochao66685247@yahoo.com, b.xiaoxu_jiangxi2021@gmail.com
***Corresponding Author**

Abstract: This paper explores the importance and methods of achieving "self-consistency" in the teaching of mathematical concepts. It first emphasizes the central role of mathematical concepts in learning mathematics and, through the example of teaching the eccentricity of an ellipse, points out the complexity of understanding mathematical concepts. Then, by exploring the history of "eccentricity," the paper demonstrates the human conventions and historical development behind mathematical concepts, highlighting the need for teachers to delve into the origins and meanings of concepts in their teaching. The paper proposes four strategies to achieve "self-consistency": sorting out the history of mathematical development, finding applications of mathematics in daily life, rebuilding connections between mathematical knowledge, and utilizing perspectives from other disciplines. Finally, the paper argues that "self-consistency" is not only a teaching method but also a teaching philosophy, aimed at helping students gain a deep understanding of mathematical concepts.

Keywords: Mathematical Concept Teaching, Self-consistency, Eccentricity, History of Mathematics, Teaching Strategies

1. Introduction

The renowned mathematician Hua Luo Geng once said: "The process of learning mathematics is a continuous process of establishing various mathematical concepts." Academician Li Bang he believes that "Mathematics is fundamentally about playing with concepts, not skills. Skills are insignificant!" The high school mathematics curriculum standards also point out: "In mathematics teaching, there should be an emphasis on understanding and mastering basic concepts and fundamental ideas. Some core concepts and fundamental ideas should be integrated throughout high school mathematics teaching to help students gradually deepen their understanding." It can be seen how important it is to deeply understand and accurately grasp mathematical concepts.

2. Are all mathematical concepts artificially defined?

It was during an open class activity for county's core teachers and their apprentices, and the topic was "the eccentricity of an ellipse." We know that the value of eccentricity (e) determines the type of conic section: ($0 < e < 1$) results in an ellipse; ($e = 1$) results in a parabola; ($e > 1$) results in a hyperbola. Eccentricity is like "DNA," determining the shape of conic sections. Below is a transcript of the teaching segment in the classroom.

Teacher: What determines the roundness of an ellipse?

Student: It should be determined by (a) and (b).

Teacher: Assuming (a) remains constant, how does the roundness of the ellipse change when (b) varies? Assuming (b) remains constant, how does the roundness of the ellipse change when (a) varies?

Student: When (a) remains constant and (b) increases, the ellipse becomes more circular; when (b) decreases, the ellipse becomes more elongated. When (b) remains constant and (a) increases, the ellipse becomes more elongated; when (a) decreases, the ellipse becomes more circular.

Teacher: Can you find a quantity to represent the roundness of the ellipse?

https://doi.org/10.62852/csa/2024/95

Student:$\frac{b}{a}$ Can represent the degree of roundness or flatness of an ellipse.$\frac{b}{a}$ The larger it is, the rounder the ellipse becomes;$\frac{b}{a}$ The smaller it is, the flatter the ellipse becomes.

Teacher: Very good,$\frac{b}{a}$ Indeed, it can represent the degree of roundness or flatness of an ellipse. Can you find any other quantity?

(Student looks puzzled)

Teacher: Can the parameter (b) be expressed in terms of (a) and (c)?

Student: $b^2=a^2-c^2$.

Teacher: $\frac{b}{a}=\frac{\sqrt{a^2-c^2}}{a}=\sqrt{1-\frac{c^2}{a^2}}$,We often use$\frac{c}{a}$ to represent the degree of roundness or flatness of an ellipse, using the term "eccentricity"$e=\frac{c}{a}$ to call it the eccentricity of the ellipse,$\frac{c}{a}$ the larger it is, the flatter the ellipse becomes,$\frac{c}{a}$ the smaller it is, the rounder the ellipse is.

Student: Why not use $\frac{b}{a}$ to represent the eccentricity of an ellipse?

(This question is indeed difficult to answer. Let's see how these two teachers deal with it.)

Teacher: Because this is a man-made regulation. If you were born in that era, you could also stipulate to use$\frac{b}{a}$ to represent the eccentricity of an ellipse.

Why is eccentricity defined as $\frac{c}{a}$, It seems that this is a trivial question, and the textbooks do not provide a clear explanation. It cannot be denied that mathematics is a product of the development of human intelligence and also a manifestation of human subjective consciousness. Various symbols, shapes, and concepts in mathematics all bear the mark of "human agency." Take the description of mathematical definitions, for example, in pursuit of logical rigor and formal perfection, it can be said that every word is carefully chosen and refined to the utmost, making the traces of "human manufacture" more apparent. Therefore, "eccentricity"$e=\frac{c}{a}$ The answer "it is man-made" seems to be fine at first glance. But upon reflection, this "man-made" does not completely dispel the doubts in the students' minds. Students might ask: "Why did humans make such a regulation? What were the considerations when it was first established?" "Is it merely a man-made regulation?" Given the importance of teaching mathematical concepts, it is essential for us to explore the truth behind the "man-made" regulations.

3. Be an archaeologist of mathematics

The author attempted to search for information on "eccentricity" and, unfortunately, did not find a detailed description of its origins and development. In Baidu Bai Ke, it describes eccentricity as follows:

Scientific Term Definition

Chinese Name: Eccentricity; English Name: Relative Eccentricity. Definition: The ratio of the eccentric distance to the radial clearance.

Applied Disciplines: Mechanical Engineering (first-level discipline); Mechanical Parts (second-level discipline); Plain Bearings (second-level discipline).

Eccentricity (Eccentricity, Eccentricity)

The given text describes the concept of eccentricity in the context of elliptical orbits and its relation to the shape of the orbit. Eccentricity is defined as the ratio of the distance between the two foci of an ellipse to the length of its major axis. This ratio is used to describe how much an elliptical orbit deviates from an ideal circular one. Orbits with a high eccentricity are more elongated, while those with a low eccentricity are closer to being circular.

The text also hints at the origin of the term eccentricity, suggesting a connection to astronomy. This is a valuable clue, but standing alone, it does not provide sufficient evidence. Further research would be needed to draw a more convincing conclusion. The text mentions the need to look for more clues in literature, but it seems that no satisfactory answers were found.

https://doi.org/10.62852/csa/2024/95

Eccentricity is indeed a term that originated in astronomy, where it describes the shape of planetary orbits. Johannes Kepler, who formulated the laws of planetary motion, used eccentricity to quantify the deviation of planetary orbits from perfect circles. The lower the eccentricity, the closer the orbit is to a circle. In the case of the Earth, for example, the eccentricity of its orbit around the Sun is relatively low, indicating that it is nearly circular.

In summary, the eccentricity of an ellipse is a measure of its deviation from a perfect circle, and it is defined as the ratio of the distance between the foci to the length of the major axis. This concept is deeply rooted in astronomy and is used to describe the shape of celestial orbits.

The formation and development of mathematical concepts have their profound real and historical backgrounds. This is undeniable, but the original generation process of many mathematical concepts has become "obscure" over time, or has gradually lost its original appearance with the development of mathematics, becoming like an "enigma" that later generations cannot figure out at all. This requires teachers to have the spirit of "archaeologists," seeking clues in seemingly unrelated information, making bold imaginations and inferences, in order to find reasonable answers. Below, the author speculates on the origin of eccentricity in conjunction with relevant clues.

Firstly, eccentricity must be related to astronomers and is widely used in astronomy. This reminds the author of the background of the emergence of conic sections. Although the theory of conic sections was established as early as the ancient Greek period, and Apollonius's "Conics" included all the properties of conic sections, making "later generations have no room to stand on," the widespread application of conic section theory should be attributed to the development of astronomy in the 16th century. Studying astronomical phenomena requires calculating the orbits of planets, which are usually ellipses of varying degrees of flatness, and eccentricity is a quantity introduced to describe the degree of flatness of the orbit. Moreover, astronomers have found that the eight major planets of the solar system all move in elliptical orbits with the sun as a focus, and the degree of deviation from the sun is also different. Therefore, they call eccentricity "eccentricity," and the distance between the planet and the sun is changing. At the perihelion, it is closest to the sun, with a deviation distance of (a - c), and at the aphelion, it is farthest from the sun, with a deviation distance of (a + c). of course, one cannot directly use the closest and farthest distances to represent eccentricity, as these values are not only related to the degree of roundness or flatness of the orbit but are also influenced by the size of the orbit. People need to construct a "stable" quantity to represent eccentricity. After repeated attempts, it was found that $\frac{a+c-(a-c)}{a+c+a+c} = \frac{2c}{2a} = \frac{c}{a}$ the value is independent of the size of the ellipse but can well characterize the degree of roundness or flatness of the ellipse. Therefore, everyone chose $\frac{c}{a}$ to represent the eccentricity.

The above speculation provides a good explanation of the origin of eccentricity from the perspective of the history of mathematical development, but there are still some shortcomings."$e = \frac{c}{a}$" Is it merely for the sake of "stability"? Surely there must be other stable quantities that can also meet the need to measure the degree of roundness or flatness of an ellipse, But why is it specifically "$\frac{c}{a}$ "? Therefore, we must continue to search for "$\frac{c}{a}$ "reasons for its rationality and scientific validity.

Starting from the definition of an ellipse, we begin our reasoning. An ellipse is the locus of points in a plane such that the sum of the distances to two fixed points is a constant (where the sum of the distances to the two fixed points is (2a), the constant is (2c), and (2a > 2c). The parameters involved in the definition are (a) and (c). Additionally, the unified definition of conic sections is "the ratio of the distance to a fixed point and the distance to a fixed line is a constant."

And the value of this constant just happens to be $\frac{c}{a}$ -From this, it can be seen that, a and c are the basic parameters that describe the definition of an ellipse and even the definition of conic sections, Therefore, using $\frac{c}{a}$ to represent the eccentricity makes even more sense.

Up to this point, the author has approached from two different angles to find "$e = \frac{c}{a}$ "three reasons for it, One from the perspective of the history of mathematics, and the other from the perspective of mathematical definitions. Of course, one could continue to try examining the rationality of eccentricity from other angles,

https://doi.org/10.62852/csa/2024/95

perhaps discovering even more reasons. For now, the "archaeology" of eccentricity can temporarily come to a close, as the aforementioned three reasons are sufficient to explain the underlying causes behind the "arbitrary stipulation."

4. The Self-Consistency in concept teaching

Perhaps some may ask: "Is your archaeological conclusion definitely correct? Was the truth exactly like this at the time?" In response, the author cannot provide a definitive answer, but at least it has achieved "self-consistency."

Baidu Encyclopedia: Self-consistency

Explanation Round: complete, comprehensive. It means that the speaker can make their arguments or lies without flaws.

When looking at the above explanation, it seems that the pejorative connotation of "self-consistency" is greater than the positive connotation. When do you need to be self-consistent? When lying? No, on the contrary, the author believes that "self-consistency" is a basic quality that teachers must have. Faced with students' thirst for knowledge and puzzled eyes, teachers must achieve "self-consistency." Imagine, if a teacher cannot be self-consistent, even if the knowledge imparted is correct and scientific, it is probably difficult to convince students; if a teacher can achieve self-consistency, even if the reasons stated have minor flaws, it can help students understand mathematical concepts. It is precisely in order to smoothly achieve self-consistency in classroom teaching that teachers need to explore teaching materials, use them creatively, consult relevant literature, and unfold imagination and reasoning. Therefore, the author believes it is necessary to give "self-consistency" a new connotation, which is: "‘Round’ the background of the emergence of mathematical concepts, ‘round’ the internal connections between knowledge, thereby promoting students' understanding of mathematical concepts."

Of course, one point needs to be emphasized, that is, do not misinterpret "self-consistency" as "fabricating." Teachers should be "archaeologists," and archaeology should be scientific and rigorous, basing on basic facts before proceeding with reasonable imagination and speculation. So, how can we specifically achieve self-consistency in the teaching of mathematical concepts?

4.1. Organize the context of mathematical history to achieve Self-Consistency

The great mathematician Poincaré once pointed out: "If we want to foresee the future of mathematics, the appropriate path is to study the history and current state of this science." History is humanity's most precious spiritual wealth. "By taking history as a mirror, one can understand gains and losses." Taking the history of mathematics as a mirror allows you to comprehend mathematics. When reading the history of mathematics, what you see is not only the light of wisdom but also the joys and sorrows, loves and hates of mathematicians. Therefore, mathematics is not "stagnant" but "vibrant." If the history of mathematics is a piece of vibrant green leaves, then those fine and crisscrossing veins are the trajectories of the formation and development of mathematical concepts. What we need to do is to patiently and carefully comb the direction of these veins to achieve self-consistency.

4.2. Find the "Shadow" of mathematics in life to achieve Self-Consistency

People often say that "mathematics originates from life," and indeed, this is true. The earliest role of mathematics was to solve practical problems in production and daily life. However, as mathematics developed, it not only solved applied problems but also brought people a great sense of achievement and satisfaction, leading to the profession of mathematicians. They study mathematics not by considering how to apply it first, but by hoping to establish a series of rules and theories to achieve perfect logical reasoning in mathematics. At this point, mathematics has risen from the initial "applied tool" to "mental gymnastics," marking that mathematics has begun to "transcend life." Once mathematics "transcends life," it inevitably faces the danger of "detaching from life," which makes it easy to understand why some mathematical concepts are so abstract and puzzling. But no matter what, the root of mathematics is still in life, and what teachers need to do is to bring mathematics back to life, to find the shadow of mathematical concepts in life, thereby achieving self-consistency.

https://doi.org/10.62852/csa/2024/95

4.3. Rebuild the connections between mathematical knowledge to achieve Self-Consistency

It is well known that various mathematical knowledge is not isolated from each other but is interconnected in many ways. For example, many new pieces of mathematical knowledge are formed and developed based on existing knowledge; the previous knowledge is the foundation of the subsequent knowledge, and the subsequent knowledge is the development of the former, thus creating the unity and continuity of mathematics. As mathematics develops, various pieces of mathematical knowledge begin to integrate. The result of this integration is the formation of numerous branches of mathematics. These mathematical branches have their own independent theoretical foundations and modes of thinking, so the gap between them begins to widen, and the boundaries become increasingly clear, and their original natural connections become less obvious and not valued. But no matter how mathematics develops, it should be a whole. In the teaching of mathematical concepts, self-consistency can be achieved by rebuilding the connections between mathematical knowledge.

4.4. Achieve Self-Consistency by standing from the unique perspective of other disciplines

All disciplines are originally interconnected and permeate each other. It is no exaggeration to say that many disciplines "were one family five hundred years ago." For example, physics and mathematics both originate from natural philosophy, and the emergence of chemistry is closely related to alchemy. Therefore, teaching any discipline cannot rely solely on the study of the discipline itself, especially for a discipline like mathematics that has a profound influence on various fields of natural and social sciences. It is necessary to try to step out of the discipline's own category and stand on the shoulders of other disciplines to achieve self-consistency.

5. Conclusion

In summary, the author believes that "self-consistency" is not only a means but also a teaching philosophy. It is a philosophy that pursues the truth of mathematical concepts with the spirit of "gradually thinning belts without regret, and withering away for the beloved."

6. References

[1] L v, Z. (2011). The importance of "self-consistency" in teaching mathematical concepts. Middle School Mathematics Teaching Reference (Upper), (3).

[2] L v, Z., & Shao, X. (2014). Cost-effectiveness analysis of four "methods" of teaching mathematical problem classes. Middle School Mathematics (Upper), (10).

https://doi.org/10.62852/csa/2024/95